Understanding the Meaning of Calculations in the Mathematics Classroom: Focusing on the Use of Tools and Inscriptions

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Abstract: I investigated the process of children's understanding in the mathematics classroom, focusing on the use of tools and inscriptions. Tools are figurative representations used to solve problems and inscriptions are figurative representations written by children and the teacher to solve and explain problems. By analyzing classroom discourse, I found that children who tried to explain the meaning of a calculation used word problems, expressions, tools, and inscriptions. This helped them to understand the meanings of the expressions. The framework used to analyze the process of learning is a "scaffolding." It is important to note that not only the teacher, but also various expressions in other children's explanations and figurative representations aided learning in the classroom. Classroom activity included whole-class and group discussions; the teacher facilitated classroom discourse using both activities. For example, at first, the children tried to explain and share their ideas in a whole-class session in which other children asked questions and discussed those ideas. When that discussion ran down, the teacher advised them to continue the discussion in groups. Subsequently, they had another whole-class session. In both activities, children referred to the printed handout showing the task and tools. I analyzed the role of tools and inscriptions written by children because the children use them to think about and explain problems. Although the teacher and children could not always share tools and inscriptions from the start, they gradually came to share them and deepened their understanding while using them. I focus on this process and try to interpret it as knowledge building in the classroom. Finally, I discuss aspects of the classroom discourse that support knowledge building.

Keywords: classroom discourse, knowledge building, mathematics education, scaffolding, tools and inscriptions

Introduction

In this study, I examined the relationship between understanding the meaning of calculations and the use of tools and inscriptions. I focused on the use of diagrams and the process of understanding in collective activity.

Current mathematics education emphasizes the role of classroom discourse to support students' learning. However, there are problems with this method. For example, some teachers who try classroom discussion have difficulty in getting the students to understand or attain the curriculum goal. Teachers must orchestrate a complex classroom discourse while supporting each student simultaneously.

Discourse and the use of tools and inscriptions

Students learning mathematics in elementary schools in Japan often use diagrams that are peculiar to the learning of Sansu, which is the primary mathematics subject in elementary school. The diagrams were developed to support students' learning and are used in textbooks. The students follow the textbook, and the diagrams in the textbook are important in helping the students to understand the learning content. The diagrams are used to solve problems, although it is often said that students cannot use them as effectively as the teacher expects. The reasons for this include a lack of spontaneity in using the diagrams, a poor choice of the diagram to use, and failure to draw the appropriate inference using the diagram. Based on such findings, Uesaka and Manalo(2006)conducted experimental classes and compared two classes: the first condition involved the active comparison of cases in which the diagrams had been used effectively and the second condition did not involve active comparison. Their results suggested that the active comparison of the diagrams and reflection on the lessons learned after problem solving promote the appropriate choice and construction of diagrams in solving math word problems.

Considering classroom discourse as a communal or collective activity, McClain and Cobb (1998)focused on the process of 'folding back' in the mathematical discourse. The notion of folding back is rooted in imagery and is consistent with Thompson's (1996) analysis, in which he argued that it is meaningful that "mathematical reasoning at all levels is firmly grounded imagery." They quote the term folding back from Pirie and Kieren's (1989) notion of folding back in psychological terms. In addition, they agree that Pirie and Kieren (1989) indicated that the actual process of learning does not proceed in a linear manner through a sequence of levels. In this way, they discuss the theoretical framework of analyzing classroom discourse, and they analyzed classroom episodes and attempted to illustrate the role of imagery and discourse in supporting the development of mathematics understanding. They clarified that the development of mathematics understanding is a recursive, nonlinear phenomenon from the viewpoint of folding back.

The roles of imagery and diagrams are important when students learn in a collective activity. In addition, the role of teachers, who support student learning and orchestrate the classroom activity, is also important. Lampert is both a mathematics teacher in an elementary school and a researcher in mathematics education. She offered a framework that unites three practice-arrows, i.e., teacher, student, and content, and consists of a triangular diagram that addresses the complexity of teaching to model the problem space in teaching (Lampert, 1998, 2001) As a practitioner, she analyzes children's learning from their activity, classroom discourse, and tools and inscriptions. She keeps a journal of her class, reflects on her own teaching and activities, and plans her students' learning.

In learning mathematics, the use of tools and inscriptions helps learners to support their problem solving and deepen their understanding. In discussing the process of learning mathematics, it is necessary to analyze the use of tools and inscriptions because learners' thoughts reflect on their use.

Support learning in the mathematics classroom

Wood, Bruner, and Ross (1976) introduced the term scaffolding in the context of adult–child interactions in which the more knowledgeable adult tutors the child to complete a task that the child would be unable to do on his or her own. Traditionally, scaffolding has been discussed in terms of one-on-one interactions. There has been little research on teacher–student scaffolding in a whole-class setting because in a large classroom, the teacher cannot interact with each student to modify the scaffolds based on all of the students' needs(Hogan & Pressley, 1997) McNeill, Lizotte, and Krajcik(2006)suggested two possible solutions. One involves putting the students in groups and then scaffolding those groups. The other involves providing students with tools, such as computers or written materials, which provide the students with scaffolds. Some previous studies provided students with tools as scaffolding (White & Frederiksen, 1998, 2000; Davis, 2003). Scardamalia and Bereiter (1991) have explored the scaffold tool, "Computer Supported Intentional Learning Environments." Bereiter et al. (2002) distinguish knowledge building from learning. Learning is an internal, unobservable process that results in changes of belief, attitude, or skill. Knowledge building results in the creation or modification of public knowledge. They designed CSILE/Knowledge Forum® to specifically support knowledge building, with provisions and scaffolding support for idea development, graphical means for viewing and reconstructing ideas from multiple perspectives, means of joining discourses across communities, and a variety of other functions that contribute to collaborative knowledge building.

I investigated the process of learning via communal activity. The tools used in the mathematics class enable scaffolding learning. Here, I focus on the role of the tool and discourse for common understanding.

Method

Data collection

I recorded lessons using a video camera and made field notes. In addition, I made transcripts of the video data. Moreover, the teacher's and children's writings could be analyzed because the teacher directed the children to write their thoughts on worksheets and he wrote the children's ideas on the blackboard to share the ideas in the classroom. These data were analyzed qualitatively.

Mathematics classroom setting

The classroom used to study the process of the learning contained 30 sixth graders. The teacher hoped that the children would acquire procedural and conceptual understanding during a classroom discussion of how to calculate the division of the fraction $\frac{2}{3}$ by 3. The main activity in the class consisted of discussion in pairs, small groups, and as a whole class. In this elementary school, Mr. Oda has been the mathematics teacher since April of this year and teaches all mathematics classes from second to sixth grade.

I interviewed Mr. Oda informally before and after each class to determine his intensions, impressions, and the goal of the class. I have participated in this math classroom continually and have watched the lessons in which the children learned about the multiplication and division of decimals and fractions since the children were fifth graders, with a different teacher. The lesson presented in this study is one of the classes that I observed.

The diagram of number-lines –suchokusenzu-

Since the fifth grade, when the children in this class learned to multiply and divide decimals and fractions, they have used a diagram of number-lines. The diagram of number-lines (Figure 1) is used as a tool to make expressions and estimate the results of calculations and the size of numbers. It consists of two number lines and a diagram that includes arrows and the sign for multiplication or division, such as ‘$\times 3$’ and ‘$\div 2.7$’, the diagram of number-relations (Figure 2). When they were in the fifth grade, they used this diagram as a tool in their discussions to explain to others and deepen their understanding of calculations. Consequently, they became used to using the diagram as a tool for thinking.

Mr. Oda rearranged the diagram of number-lines because he believes that it is difficult for his students to think with it. He changed one of the number lines to a square to denote area. He printed the diagram of line-and-area (Figure 3) on a worksheet, which contained blank areas where the students could write their own ideas during the discussion.

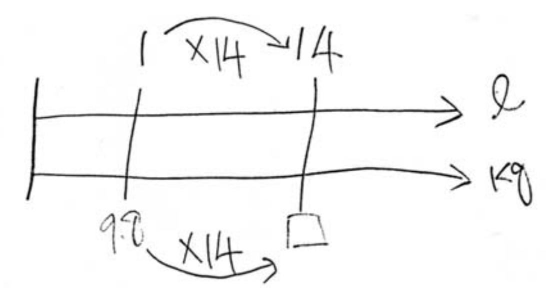


Figure 1: Example of the diagram of number-lines

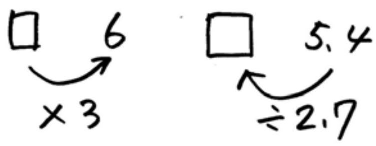


Figure 2: *Example of the diagram of number-relationship*

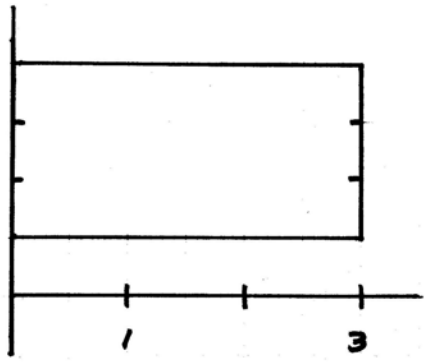


Figure 3: *The diagram of number-and-area rearranged by Mr. Oda*

Classroom episode

Initially, the children appeared to have sufficient understanding to use the diagram. Nevertheless, through the discussion, it became clear that they did not understand the diagram because they did not know what the components of the diagram meant. Mr. Oda noticed that the children needed to understand the diagram to have a common understanding for explaining things to each other. In doing so, the diagram became an intermediary when they thought about the relationship between the procedure and the meaning of the calculation.

Children's methods of explanation

The children who made an expression from the word problem explained each of their ideas. In this class, some of the children had already learned the procedure used for the calculation in the cram schools (gakusyu-juku in Japanese) that many of the children in this class attend after school. They knew the procedure as "invert and multiply." However, Mr. Oda said that the explanation that they had learned in cram school was not good enough. He asked the students to explain why such a calculation procedure was possible. Moreover, he wanted all the students to be able to understand the explanation. The ideas that the children came up with included wrong answers. Consequently, in the class, they gradually began to discuss two ideas concerning correct answers.

Many of children insisted on Idea A (in Table 1), especially the children who had already learned the procedure. But they could not explain it sufficiently, for the focus of the discussion moved to Idea B (in Table 1). And then the group discussion began. Mr. Oda's goal was to obtain procedural understanding involving an understanding of the meaning, not the formal understanding that children often show concerning the division of fractions. He asked the children to explain the process of the calculation. He found that the children read the word problem, devised an expression, and calculated the result.

Table 1: *The Word problem and Ideas of calculations*

word problem	We can sterilize a flower garden of $\frac{2}{3}$ ㎡ with 3 dℓ. How much ㎡ apply this medicine with 1 dℓ?	
ideas	Idea A	Idea B
calculations	$\frac{2}{3} \div 3$ $= \frac{2}{3} \div \frac{3}{1} = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$	$\frac{2}{3} \div 3$ $= \frac{2 \div 3 \times 3}{3 \times 3} = \frac{2}{9}$

As Mr. Oda had anticipated, the children could not keep the discussion going. Initially, they tried to explain the procedure using the expression only. They did not refer to the word problem or a diagram of number-lines that they had used previously. In the whole-class discussion, the students tended to think alone before telling everyone their idea. Mr. Oda then had the students discuss the problem in groups. The children then gradually began to talk more about their ideas.

In the group discussions, the children could not explain their own idea well and their understanding of the problem seemed vague. In the whole-class discussion, it became clear that they did not understand the diagram. The main topic of the discussion moved to "Where is 1?" and then to "What does 1 mean?" When the teacher told the children to think with the diagram, they found that their understanding seemed to change as did the way they used the diagram.

The children's understanding of the components of the diagram was poor. As a result, they did not share the diagram as a common tool. In addition, the children did not think about the problem situation. Moreover, they did not seem to think by operating the expression.

1. Kinoshita's initial explanation

In Taiyo's group discussion

Taiyo: To multiply '÷3' by 3 is "the same answer"

Others: Oh, yes.

Taiyo meant that doing the calculation [$\div 3 \times 3 = 1$] did not change the answer to this problem. Another child in the classroom used the same explanation that Taiyo did. Therefore, Mr. Oda asked what meaning the denominator took [3×3], but the children could not explain this. At this point, Mr. Oda suggested that the children explain it using the diagram on the worksheet that he had given each student at the beginning of this class. Then, several children raised their hands. Mr. Oda told Kinoshita to explain her idea. Kinoshita pointed at the diagram that Mr. Oda had printed on the blackboard and began her explanation (Figure 4).

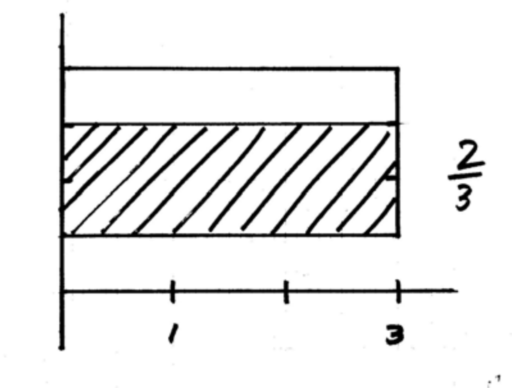


Figure 4: *The diagram of line-and-areas on the blackboard (1)*

Kinoshita*: That was 1, the place that was 1 becomes 9.*
Mr. Oda*: Where is 1?*
Kinoshita*: Here*
Mr. Oda*: What does Kinoshita mean by 1? 1 what?*
Children*: 1 dℓ!*
Children*:1 m^2!*
Mr. Oda*: Which is correct? Please discuss this in your groups.*

When the discussion dwindled, Mr. Oda used the diagram to suggest what the children should think about. For a while, the discussion became more animated. Nevertheless, it became clear that the children's understanding of the diagram was insufficient. The other children did not appear to understand Kinoshita's explanation. Therefore, Mr. Oda asked the children what each component of the diagram meant, so that he illustrated the vagueness of their understanding indirectly. Then, the children began to talk about the meaning of the diagram.

In the group discussion, Taiyo, who considered the area in the diagram to represent 1 m^2, thought of an explanation that used the diagram. Before beginning the group discussion, he compared it with the diagram of number-lines that he had used. Taiyo was able to confirm what quantity the diagram with two number lines showed by using an illustration of this number line. However, he still thought of a way to explain things.

When they learned to multiply and divide decimals, they used the diagram of number-lines. Taiyo used this, but did not find it useful, just as the teacher expected, although it is used in textbooks as a tool for thinking and making expressions. Because they had practiced using it, they were accustomed to using it. Nevertheless, it did not help Taiyo.

Before they could discuss the problem using the diagram, the children needed to be able to share the diagram as a common tool. It was necessary for the children to understand it, and to share their understanding. During the discussion, they exchanged their ideas. Some referred to the problem situation, whereas others referred to the units in the word problem. Like Taiyo, some of the children referred to another tool or inscription that they had written. Because there are various ways to solve a problem, discussion enables the exchange of the various ways of thinking to promote understanding.

Gradually, the children began to understand the problem and shared ideas, and the teacher confirmed what they understood and shared.

2. Understanding the details of the diagram

Mr. Oda judged that the children would not understand the diagram of number-lines consisting of two lines. The word problem involved the area of a flower garden. Therefore, Mr. Oda replaced the diagram of number-lines with a line-and-area diagram. As he expected, the children could not use the diagram of number-lines as a tool to solve this problem, although they used it as a tool when they devised an expression to solve the problem. At this point, Taiyo wrote a multiplication sign with an arrow on the diagram and he traced the line-and-area diagram with a pencil as he thought. The teacher considered the state of the group discussion and reverted to a whole-class discussion.

Amano*: I think that the upper (diagram of an area) means an area because "dℓ" is written on the lower (number line). I think that the upper diagram means "m^2". It is difficult to read if a dial disappears or the bottom becomes a square meter.*

Mr. Oda*: Taiyo, what did you do?*

Taiyo*: I used the diagram of number-lines.*

Mr. Oda*: You tried expressing this word problem using a diagram of number-lines, didn't you? Did you find anything? Ok? What do you think the bottom line means if you express it in the diagram of number-lines?*

Amano*: The lower line means dℓ, the top one means m2. I think.*

Mr. Oda*: What does the lower line become?*

Children*: It is the thing that is 1!*

Mr. Oda*: It is the one that has 1, isn't it? Well, I will put 1 on the lower line, and I will put in the number that we know. Where do we put the ⅔? In the diagram of* *number-lines, ⅔ goes here (on the lower line). Where does it go in the diagram of the area?*

Nakazawa pointed at the whole diagram of the area, and then the other children looked at it silently for a while. Some of the children said that it differed from their idea. Next, Saeki circled the right part. Although the other children initially said nothing on seeing this, Usugi said that his idea was the same as Saeki's. Subsequently, Mr. Oda asked the children what the area that Nakazawa pointed to meant. Then, the children, who were initially silent, began to mention their ideas. They said that the area Nakazawa pointed to meant "1." The content that they checked in the diagram of number-lines was re-confirmed by comparing the units and number value with the word problem.

3. Kinoshita's second explanation

During this classroom discussion, the children shared their understanding of the diagrams using the meaning that each person had captured intuitively. In this way, they returned to the theme that they discussed earlier, "what is 1?" after having shared their understanding of the details of the diagrams. Mr. Oda confirmed the meaning of the details of the diagram and let all the children in the classroom share it. Consequently, the children came to have a detailed common understanding about the meaning of the diagram. The children compared the diagram with the situation expressed in the word problem and the position where the units and the amount are used to deepen their understanding.

Mr. Oda*: Can you understand where 1 is? It means "1 m^2," doesn't it? With this 1, one, two, three, 1 is divided into three pieces, and two of the pieces become '⅔,' don't they? Is it good?*

After confirming what '⅔' meant, Mr. Oda again asked the children what '⅔' meant in the numerical formula that they had discussed at the beginning of a class.

Taiyo*: When '⅔' is divided into three pieces equally, it becomes the part that is surrounded by the yellow line.*

Mr. Oda*: Do you all understand? Should Taiyo explain this once more?*

Taiyo*: We divided it into three equal pieces, and '⅔' is two of the pieces: one, two, and three.*

Mr. Oda*: Okay. Then, let' s go back to the expression once again. What does this* 3×3 *mean?*

Kinoshita*: I divided 1 m^2 into nine pieces, or* 3×3.

They understood how 1 square meter was expressed in the diagram and confirmed ⅔ m2 in the diagram. Mr. Oda showed them that '⅔' is two of the three parts into which the area is divided (Figure 5). Then, Taiyo used the phrase, "divided into three equally" to explain '$\div 3$'

in a numerical formula. With Taiyo's explanation, the other children nodded and his expression implied agreement. The children discussed this, and Mr. Oda presented the problem being discussed again. Subsequently, Kinoshita explained it in the same manner as in the initial discussion. The words she used in her explanation were essentially the same as in her initial explanation.

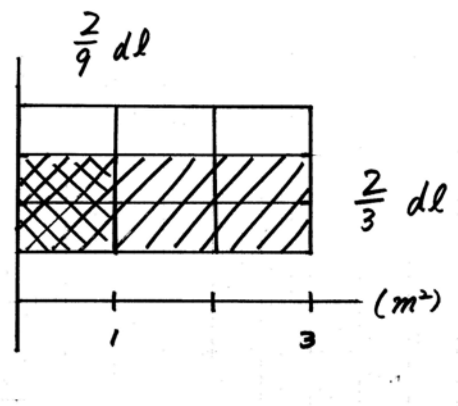


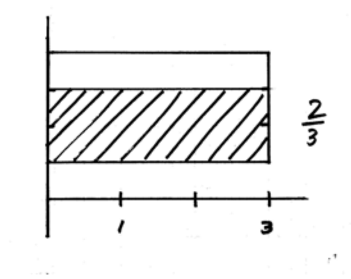
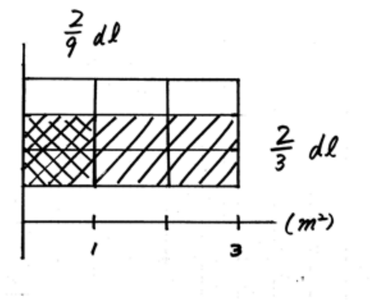
Figure 5: *The diagram of line-and-area on the blackboard (2)*

However, the children agreed with this explanation after their discussion. During this discussion, Mr. Oda judged whether the children had any questions and what children did not understand and he asked the children what they want repeated and showed it clearly in the diagram. In addition, he changed the theme of the discussion at that time and articulated the problem that they should discuss.

After the diagram became a common tool in this class, the teacher returned to the previous problems. The children could confirm what 1 meant, where 1 was, and what $\frac{2}{3}$ meant in the diagram by referring to the problem. Finally, they could understand the meaning of the calculation because they could refer to the expression with the problem situation. In addition, at the beginning of this lesson, they could not explain Idea A in Table 1, but After this discussion they could understand it. To understanding the expressions used in the calculation, they used their familiar knowledge which they had acquired referring to the diagram which is their common tool (show Table 2). 'Expressions' in Table 2 is the process of the calculations. 'Operations' is the parts. 'Students' knowledge about the operation' is the way of the explanation of the operation. In this way of Table 2 about the calculation, students' knowledge, and the figurative representation used in this lesson, I found that the process of changing the use of the tools and inscription was similar to the process of calculation which students wrote before discussion.

Table 2: *Expressions and students' knowledge*

	knowledge about division of fraction; 'invert and multiple'
	$\frac{2}{3} \div 3$ $\frac{2}{3} \div \frac{3}{1}$ $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

expressions			
operations	a) ⅔ b) ÷3	(÷) $\frac{3}{1}$	a) $\times\frac{1}{3}$ b) $\frac{2}{9}$
students' knowledge about the operation	a) two pieces of the things divided into three b) to divide one into three pieces'	In changing integer to a fraction, we put 1 to a denominator.	a)to have one of the things divided into three b)two pieces of the thing devided into nine, which is divided into three, and again divided into three.
the diagram to solve this problem.			

Discussion

I described a case in which the children came to accept one girl's explanation during a discussion. I discuss how the classroom discourse using the diagram supported the children's understanding as follows:

1. Using the diagram gave the teacher and children a common tool for explaining their ideas to each other.
2. It was necessary for the children to understand the diagram sufficiently to use it effectively.
3. Sharing the diagram as a tool helped the children to exchange their ideas.

In this case, the process of learning can be seen as the "folding back" of Cobb et al. (1998). When trying to solve the problem at the beginning of this lesson, there was a lack of understanding of how to use the diagram. The activity of explaining this to others enabled them to discover how to use the diagram. From the view of folding back, Kawano(2005) clarified the mathematical tools used in mathematical learning played the role of imagery of the task, and not only supported the children's development of conceptual understanding but also developed the understanding of the mathematical tool itself in using them repeatedly. In addition, she indicated that both developments mutually promoted the understanding by analyzing the process of the lessons. In this study, the understanding of the diagram developed that of the meaning pf calculation which is abstract.

When a child explained what he/she understood intuitively to someone else, he/she had to provide an explanation that could be understood. In the discussions in pairs or small groups or with others who had the same idea, a vague explanation was often accepted and it seemed to make their idea clear. By contrast, in the whole-class discussion or when they had different ideas, a much more logical explanation was required. The children could not conduct much of a discussion when their understanding was vague. The discussion clarified what the children did not understand and could not understand sufficiently.

As well as the activity involved in the explanation, the teacher clarified the vagueness of the children's understanding through classroom discussion. He gave the children a diagram that was arranged so that it could be used easily. The diagram played an important role, and the written explanation, in addition to the spoken language, helped the children in this classroom. The diagram was the mathematical representation of the problem, and it is a more concrete representation of an abstract numerical formula. It became the intermediary between the concrete problem and the abstract expression of the calculation.

As Thompson (1996) notes, once students begin to solve a problem, they tend to decouple their activity from the problem situation. Therefore, the teacher needs to encourage the children to develop "imagery" of the problem situation and use it to solve the task. When the children were given the diagram, they tried to understand the relationship between the diagram and the problem.

The teacher provided the children with the tool for thinking and led the classroom discourse to support their understanding. After he provided a setting in which the children could have a discussion, he articulated the topic that they should discuss and facilitated the discussion style, i.e., in groups or as the entire class. He judged their understanding and the learning process and facilitated the activity consistently, and scaffolded the children.

His scaffolding was not one-to-one, but the children made use of the explanations of other children and deepened their understanding during the discussion. In addition, one-to-one scaffolding occurred among the children, and the teacher facilitated this. Although it is difficult to construct scaffoldings in a classroom, the scaffolding in this case shows important aspects of the constitution and facilitation of the lesson.

When it comes to scaffolding, fading is an essential characteristic. Tabak (2004) stated that scaffolds serve to help learners complete a task independently and they should fade as the learners develop their own understanding in comparison with "a cultural tool." In the classroom, the diagram was used as a tool to support learning. It helped the children to understand the calculation when the teacher wanted them to acquire an understanding of the meaning. Nevertheless, in the future they will not always use the diagram. Children may use a diagram when they feel that they need it or forget how to do the calculation. Indeed, it was not the only scaffolding given to children because they could understand the diagram in the discussions. Using, understanding, and sharing the diagram supported the children's learning. In addition, the teacher facilitated the learning or the use of the diagram.

Finally, I discuss the knowledge building in this class. The teacher did not aim at knowledge building, but he intended to connect the ideas suggested by himself and the children. The other teachers in this school try to practice "manabiau" (learning together) to build the learning community. The terms "manabiau" and "tsunagu" (connect the ideas of children) are important for teachers engaged in school improvement in Japan. In the classroom of this study, the children explained their ideas to each other, which deepened their understanding. By repeating this process, they came to a common understanding and shared it. Some of the children already had ideas about the calculation or developed them intuitively. In exchanging and explaining their ideas, they reached a common understanding This process can be seen as knowledge building because the teacher and the students were learning together in the activity of explain to others and tried to understand each other and use their explanation for understanding. In addition, this study shows how the teacher gave the scaffolds the students in the whole class discussion and enabled the discussion and the use of figurative representations to gave the scaffolds by analyzing the classroom discourse, including the use of tools and inscriptions.

The present study was carried out in one class of fifth grade. Future investigations will need to examine in other classes and grades. In addition, the more case studies in more detail is needed to build on theoretical framework to analyze the process of learning and knowledge building in the term of Bereiter et al and to support the classroom practice.

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References

Davis, E. A. (2003). Promoting middle school students for productive reflection: Generic and directed prompts. *The Journal of the Learning Sciences, 12*, 91-142.

Hogan, K. & Pressley, M. (2001). *Scaffolding students learning: Instructional approaches and issues*. Cambridge, MA: Brookline Books.

Kawano, M. (2005) Jugyou ni okeru "suugaku tool no shiyou to gainen rikai no kentou [Use of a "mathematical tool" and Conceptual Understanding in a Classroom -Focusing on P.Cobb's "folding back"-. *Research Journal of Educational Meathod, 13*. 13-25. Japan.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* NY :Yale University Press.

Lampert, M. (1998). Investigating Teaching Practice. In Lampert, M & Blunk, M. L. (Eds). *Talking Mathematics in school;Studies of Teaching and Learning*. (pp.153-162).

McClain, K. & Cobb, P. (1998). The role and discourse in supporting Student's mathematical development. In Magdalene Lampert and Merrie L. Blunk (Eds.) *Talking mathematics in school: Studies of Teaching and Learning.* (pp. 56-81), NY: Cambridge University Press

McNeill, k. L., Lizotte, D.J., & Krajcik, J. (2006). Supporting Students' Construction of Scientific Explanations by Fading Scaffolds in Instructional Materials. The Journal of the Learning Sciences., 15(2). 153-191.

S. Pirie& T. Kieren(1989). A recursive theory of mathematical understanding. *For the learning of mathematics, 9(3)*, 7-11

Scardamalia, M., & Bereiter, C. (1991). Higher levels of agency for children in knowledge building: A challenge for the design of new knowledge media. *The Journal of the Learning Sciences, 1*, 37-68.

Scardamalia, M., & Bereiter, C. (2002). Knowledge Building. In Encyclopedia of 4

Tabak, I. (2004). Synergy: A complement to emerging patterns in distributed scaffolding. *The Journal of the Learning Sciences, 13*, 305-335.

Thompson, P. W. (1996) Imagery and the Development of Mathematical Reasoning. In Leslie P. Steffe et al. (Eds). *Theory of Mathematical Learning*.(pp.267-284). Mahwah, NJ: Erlbaum Associates.

Uesaka, Y. & Manalo,E. (2006). Active Comparison as a Means of Promoting the Development of Abstract Conditional Knowlege and Appropriate Choice of Diagrams in Math Word Problem Solving, In Dave Barker- Plummer et al. (Eds).*Diagrammatic Representation and Inference, Lecture Note in Artifical Intelligence, Vol 4045*. 181-195.

White, B., & Frederiksen, J. (1998). Inquiry modeling, and metacognition: making Science accessible to all students. Cognition and Instruction, 16, 3-118

White, B., & Frederiksen, J. (2000). Metacognitive facilitation: An approach to making scientific inquiry accessible to all. In J. Minstrell & E.v. Zee (Eds.), *Inquiry into inquiry learning and teaching in science.* (pp.331-370). Washington, DC. American Association for the Advancement of Science.

Wood, D., Bluner, J. S. & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*. 89- 100.